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Calculation of Turbulent-Inviscid Flow Interactions with Large Normal Pressure Gradients

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A simple and economical iterative scheme is presented for calculating turbulent shear layers with significant pressure gradients normal to the plane of the layer, such as occur on highly-curved surfaces or near the trailing edges of lifting airfoils, and for matching the shear-layer calculations to calculations of the inviscid external flow. The iteration required to solve the elliptic equations describing the shear layer is combined with the iteration needed for the matching, and the finite-difference solution of the normal-component momentum equation is a simple quadrature at each iteration. Therefore computing time is little greater than in conventional displacement-surface calculations that ignore normal pressure gradients.

Introduction

BOUNDARY-layer (thin-shear-layer) equations are derived from the Navier-Stokes equations by assuming that streamwise (x) gradients of velocity are small compared to velocity gradients normal to the surface (y). The resulting simplifications[1] include the disappearance of the pressure gradient normal to the surface, from which follows the smallness of the velocity gradient $\partial U/\partial y$ in the "inviscid" flow just outside the shear layer. This in turn leads to the concept of a displacement thickness to represent the displacement, nominally independent of y, of the "inviscid" flow streamlines near the shear layer. It is important to note that once the basic assumption fails, *all* the simplifications disappear. If the shear layer changes rapidly in the x direction, so that $\partial V/\partial x$ is large, normal pressure gradients are important both within the layer and outside it; not only does the displacement thickness fail to represent the displacement of the external flow, but its definition, even as a mere shear-layer parameter, becomes ambiguous. It will be seen that corrections based on the surface radius of curvature may be highly inaccurate.

All but the most rapidly changing viscous or turbulent flows will still be recognizable as fairly thin shear layers, and the terms in the Navier-Stokes equations that are neglected in the boundary-layer equations will still be fairly small. Therefore, thin-shear-layer concepts are still useful in analytic and computational work. In particular, the effects of the normal pressure gradient on the shear layer will be small enough to be included by iterative improvement of a conventional marching calculation rather than by a fully elliptic calculation, and this more economical approach has been adopted in the present work.

Analysis

The Navier-Stokes equations for a fluid with viscous and/or turbulent stresses $\sigma_{ij}=\sigma_{ji}$ can be written, for two-dimensional flow, as

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}\right) \quad (1)$$

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}\right) \quad (2)$$

with the continuity equation for incompressible flow

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \quad (3)$$

In the boundary-layer equations, Eq. (2) collapses to $\partial p/\partial y=0$ and the gradient of σ_{xx} in Eq. (1) is neglected. In the present work we have retained Eq. (2) but have assumed that all the stress-gradient terms except the gradient of σ_{xy} in Eq. (1) will remain small enough for fairly crude approximations—both physical and numerical—to suffice. Specifically, for turbulent flow we calculate $\sigma_{xy}=-\rho\overline{uv}$ by means of a turbulence model and represent the other stresses by empirical multiples of σ_{xy}. However, the solution procedure could also be used with a full-stress equation turbulence model, or even an eddy-viscosity model.

The calculations are carried out in semi-curvilinear (s,n) coordinates,[2] the s axis being coincident with the (curved) surface, or, in the case of a free shear layer, with a nominally defined centerline; the constant s lines are straight and normal to the s axis. For simplicity, however, the description will be presented in x,y coordinates. The boundary conditions for this elliptic system of equations, Eqs. (1-3) plus the turbulence model[3] are, for the boundary-layer case, as follows: 1) p specified for $0<y<\delta$ at $x=0$ (initial) and $x=x_{max}$ (final); the edge pressure p_e specified for $y=\delta$ at all x, thus specifying the edge velocity U_e via Bernoulli's equation as usual, if V_e is known; 2) the conventional boundary-layer conditions: U specified for $0<y<\delta$ at $x=0$; $U=0$, $V=V_w$ for $y=0$ at all x (or analogous conditions for a wake). Usually, $V_w=0$. The specification of p at $y=\delta$ could be replaced by the specification of V, as in internal flows: in free shear layers p would be specified at both edges of the shear layer and there would no longer be a boundary condition on V. The present routine can easily be reformulated to accept these other boundary conditions.

The shear-layer calculation loop for the boundary-layer case, assuming $p_e(x)\equiv p(x,y=\delta)$ is known and fixed, is:

1) Guess $p(x,y)$, by equating $\partial p/\partial y$ to $\rho U_0^2/R_w$, where $1/R_w$ is the local surface curvature and $U_0(y)$ is the velocity at $x=0$, or otherwise.

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2) Calculate U and the σ stresses at the end of one streamwise step, as in a conventional boundary-layer calculation, but with $\partial p/\partial x$ as a ("known") function of y, evaluated from the guessed $p(x,y)$ by central differences in x.

3) Calculate V at the end of the step from the ordinary differential equation that results from substituting Eq. (3) into Eq. (1) when U, p, and the σ stresses are "known" at all y, or otherwise, again as in a normal boundary-layer calculation.

4) Calculate p at the end of the step by integrating Eq. (2) downward from $y=\delta$, where p_e is specified by step 1 (U, V and the σ stresses now being taken as "known" at all y).

5) Repeat steps 2-4, always evaluating $\partial p/\partial x$ by a central-difference formula from the previous guess (step 1) for $p(x,y)$ and *not* from newly calculated values, until $x=x_{\max}$. In the terminology of numerical analysis, this is a Gauss-Jacobi iteration: the Gauss-Seidel strategy of using newly calculated p values as soon as available would lead to gross errors in $\partial p/\partial x$.

6) Repeat steps 2-5 for a new sweep from $x=0$ to $x_{\max}$, now using the values of $p(x,y)$ calculated in the last sweep: continue until converged values of surface pressure, say, are obtained.

Only p has to be stored as a two-dimensional array in (x,y); storage is otherwise the same as for a conventional boundary-layer calculation. Also, the time spent on the simple quadrature, step 4, used to obtain p is very small. In our calculations, using the numerical method of Bradshaw et al.[3,4] the pressure calculation is simply an extra subroutine inserted with little disturbance to the existing program, and the equations used in steps 2 and 3 are still a hyperbolic set solved by the method of characteristics. If $\partial p/\partial y$ is expected to be negligibly small the pressure calculation could be bypassed and $\partial p/\partial x$ equated to its value at $y=\delta$ as usual. As in the matching method of Ref. 5, two-dimensional storage for p would be needed only in the region of significant $\partial p/\partial y$, which may be of quite small streamwise extent in, say, the flow over the knee of an airfoil flap. The strategy of the calculations is similar to the "partially parabolic" method of Pratap and Spalding[6] for internal flows, but the latter method is completely different in detail. The upstream influence present in any elliptic system proceeds, at a rate of one x step per sweep, via the use of centered differences for $\partial p/\partial x$, much as in a conventional field solution of say, Laplace's equation with a 5-point centered-difference molecule: $\partial p/\partial x$ at station x on sweep n contains p calculated at station $x+\Delta x$ on sweep $(n-1)$. Clearly this numerical scheme can propagate only the upstream influence due to pressure disturbances and not that due to mean-flow transport in a reversed flow. One may expect that a necessary condition for stability will be the smallness (not negligibility) of upstream influence due to the x-wise stress gradients, but one may also expect this condition to be satisfied in almost all shear layers.

With the boundary conditions on the pressure fixed, the calculation, like a conventional boundary-layer calculation, is unrealistic, since it covers only part of the flow. Matching to an external-flow calculation is, however, very simple, involving no "integral" thicknesses and no approximations except discretization errors. The shear-layer calculation with given edge pressure $p_e(x)$ yields values of $V_e(x)$ at $y=\delta$, and Laplace's equation for the inviscid flow outside the contour $y=\delta(x)$ can then be solved with $V_e(x)$ as a boundary condition to yield new values of $p_e(x)$. The complete calculation thus proceeds as follows:

A) Make a generous guess of $\delta(x)$, $\delta_0(x)$ say, estimate $V_e(x)$, and solve the inviscid-flow equations outside $y=\delta_0(x)$ to get $p_e(x)$.

B) Execute steps (1-5) that is, one sweep of the shear-layer calculation, inside $y=\delta_0(x)$, to obtain new values of $V_e(x)$.

C) Solve the inviscid-flow equations outside the *same* boundary $y=\delta_0(x)$ with the new values of $V_e(x)$ to obtain new values of $p_e(x)$. Deduce new values of $p(x,y)$ for $y<\delta_0$ by assuming that $\partial p/\partial y$ remains the same as that calculated in B.

D) Repeat steps B and C until convergence is achieved. Note that upstream influence due to pressure disturbance now also propagates via the inviscid flow.

At present $\delta_0(x)$ is revised, after the first sweep only, to equal 1.1 times the actual shear-layer thickness calculated on the first sweep. Thus, the first guess of δ_0 can be very crude. In any case it is not critical because the shear-layer calculation can be extended into the external stream with no difficulty except a waste of computing time, at least as long as $d\delta_0/dx$ remains small. Also, if the true boundary-layer thickness δ exceeds δ_0 a small correction to the calculated V at $y=\delta_0$ could be made to allow for the small total-pressure deficit in the region $\delta_0<y<\delta$. The advantage of keeping a constant contour $y=\delta_0(x)$ for the inviscid-flow calculation is that the mapping of that contour in a conformal-transformation method, or the inversion of the influence matrix in a surface-singularity method such as that used here, need not be repeated at each sweep.

Some care is needed in the discrete representation of p_e and V_e, because the shear-layer and inviscid solutions depend on the streamwise derivatives of these quantities. Some smoothing of the calculated values is advisable, especially when the calculation stations for the inner and outer flows do not coincide. The problem is found in conventional (displacement-thickness) matching also, but it is likely to be more severe in the present, rapidly changing flows. At present, a sufficient number of pressure profiles is chosen from the results for all the x steps of the shear-layer calculation, and cubic splines are fitted in the x direction at each y; on the next sweep the spline fits are interpolated in the x direction to provide input values of $\partial p/\partial x(y)$ for each shear-layer calculation station. This removes any short-wavelength noise resulting from irregular input data. In rapidly changing flows the spline nodes are best chosen by the operator (or by the computer) after inspection of the output of each sweep, so that sharp peaks in $p(x)$ are not incorrectly smoothed out. The calculation loop is shown in Fig. 1.

A sweep-to-sweep under-relaxation factor of 0.3 is applied to the values of $V_e(x)$ calculated from the shear-layer program and to the values of $p-p_e$ calculated from Eq. (2). No attempt has yet been made to optimize the under-relaxation factors. Convergence is usually monotonic after the first sweep, which suggests that the under-relaxation factors could be increased for later sweeps. However, even the sharp bend calculation shown in Fig. 2 converged in ten sweeps (Fig. 3) starting from the very crude guess

$$\partial p/\partial y = 0 \tag{4}$$

for the pressure gradient within the shear layer, so that there is little chance of a large improvement. It should be noted that the numerical problems of instability or false diffusion in conventional Navier-Stokes calculations are effectively absent from the present shear-layer calculations because the in-

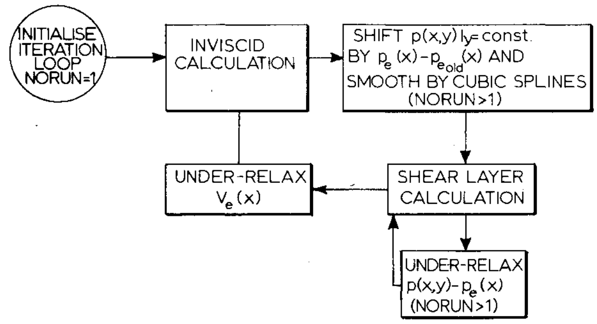

Fig. 1 Flow chart of viscous-inviscid interaction calculation. "NORUN" is iteration count.

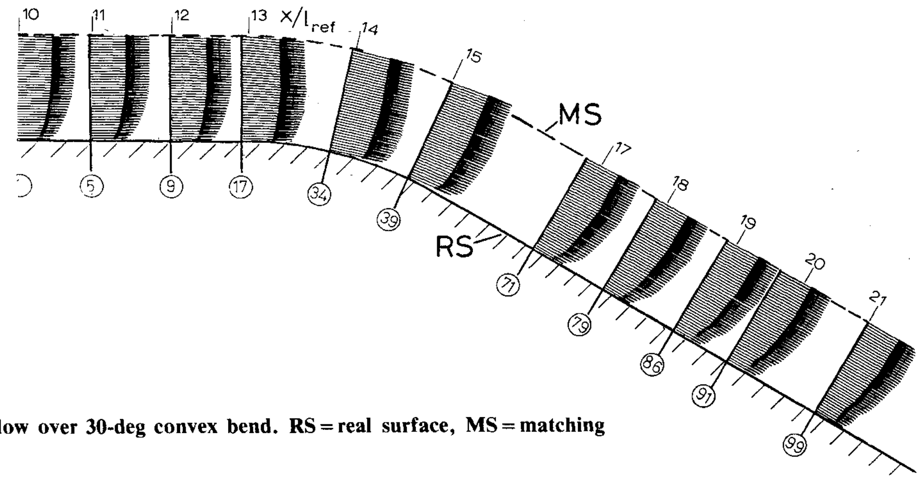

Fig. 2 Calculated streamline directions in flow over 30-deg convex bend. RS = real surface, MS = matching surface. Figures are x-step counts.

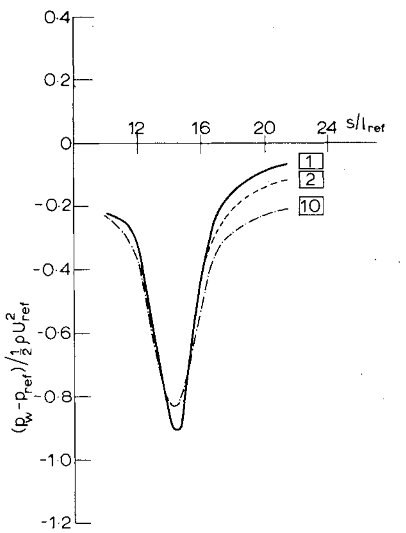

Fig. 3 Convergence of wall pressure in calculation for 30-deg convex bend. Curves are outputs after the number of iterations indicated.

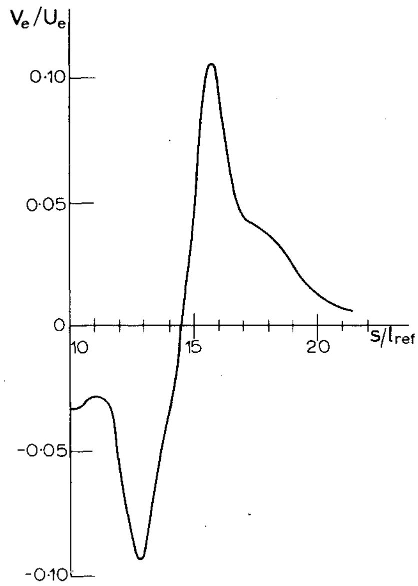

Fig. 4 Calculated streamline angle at matching surface in flow over 30 deg convex bend (see Fig. 2).

clination of the streamline to one set of coordinate lines is fairly small. (The method of characteristics used with the Bradshaw et al. model uses the Courant x-step condition to insure stability, so that the main effect of large streamline inclinations is simply to increase the computing time.)

For the typical case of an airfoil with small regions of large normal pressure gradient (near the knee of a simple flap or slat, and near the trailing edge) the above calculation takes very little more time, and not much more storage, than a conventional viscous-inviscid calculation with matching via a displacement thickness, because the solution of Eq. (2) for p can simply be bypassed if $\partial p/\partial y$ is negligibly small. As in that case, core storage can be saved by transferring the shear layer and inviscid sections of the program to and from disk storage.

The addition of the pressure-calculation routine to the Bradshaw-Ferriss-Atwell numerical method required rather less time and effort than conversion from x,y to s,n coordinates. At present, the inviscid (incompressible) calculation uses the Smith-Hess method.[7] Either external or internal flows can be handled. One complete sweep of the viscous/inviscid calculations, consisting of: 1) 100 streamwise stations and 39 points across the shear layer (for the shear layer calculation); 2) 51 points on the body contour (for the inviscid calculation), takes about 25 s on the CDC 6500 computer at Imperial College, excluding the inversion of the influence matrix for the inviscid calculation (which takes about 7 s). Most of the time is taken by the shear layer calculation. The smoothing of the two-dimensional pressure array (100×39) using cubic spline interpolation takes about another 9 s per sweep.

In many cases a much smaller number of shear-layer steps would suffice. The maximum amount of core needed (for the 51-point inviscid calculation) is 25,000 decimal words. The present calculations are now being extended to the wake; the shear layer calculation method itself has been successfully used in a wake by Morel and Torda[8] and by Huffman and Ng.[9]

Results

Sample calculations are shown in Figs. 2-8. Figures 2-7 show the internal flow investigated by Smits et al.[10] in which a two-dimensional constant-pressure boundary layer on the wall of a duct encounters a convex bend whose radius R is about five times the boundary layer thickness. The duct's height is the same as the bend radius, and the bend angle is 30 deg. Figure 2 shows the velocity vectors, which are far from parallel to the surface. Figure 4 shows the streamline angle at the nominal $y = \delta$, and Fig. 5 shows that, as a consequence, the pressure gradient normal to the boundary layer is far from the value predicted by the "centrifugal" formula

$$\frac{\partial p}{\partial y} = \frac{\rho U_e^2}{R_w} \tag{5}$$

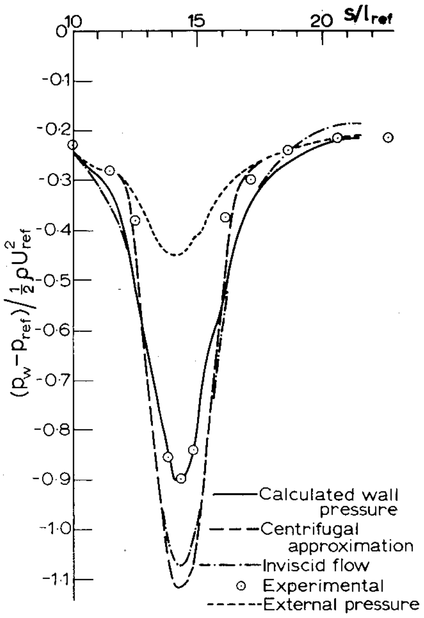

Fig. 5 Pressure distribution in flow over 30-deg convex bend. ⊙ **measured wall pressure; ——— calculated wall pressure; ----- calculated pressure on matching surface; — · — · — wall pressure from wholly-inviscid calculation; ———— wall pressure deduced from calculated pressure on matching surface and the "centrifugal" formula $\partial p/\partial y = \rho u_e^2/R$, where u_e is velocity at matching surface and R is surface radius of curvature.**

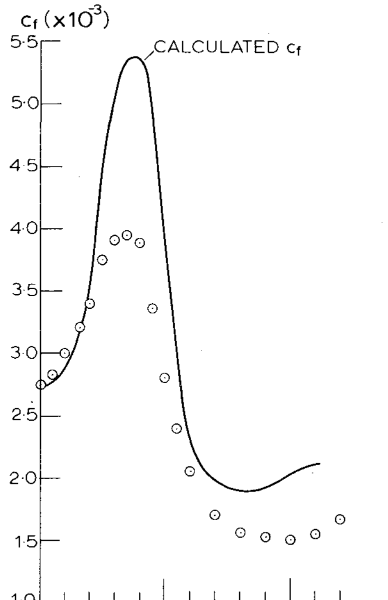

Fig. 6 Distribution of skin-friction coefficient on 30-deg convex bend. ⊙, **measurements; ———, calculations (without allowance for stabilizing effect of streamline curvature).**

where R_w is the surface radius of curvature, which is often suggested as an adequate approximation for mildly curved flows. The reason, of course, is that in the flow of Fig. 2 R_w and p change rapidly with x. The turbulence model used in the calculations contained no allowance for the effect of curvature on the turbulence structure—development of such allowances for highly-curved flows is in progress—and the calculated skin friction (Fig. 6) does not agree very well with experiment. It is generally higher, as one would expect from the stabilizing effect of convex curvature, although finite-difference errors due to large $\partial \sigma_{xy}/\partial y$ may be present in the

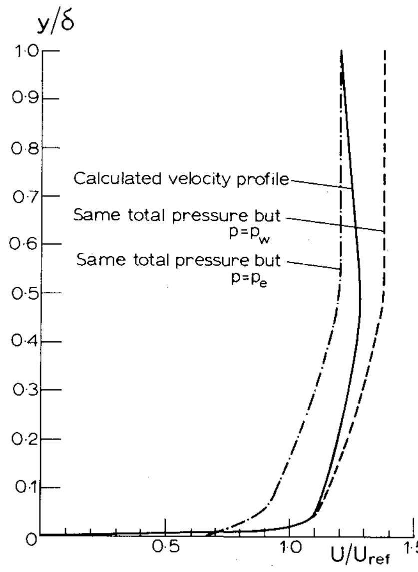

Fig. 7 "Velocity" profiles calculated at $x/\ell_{ref} \simeq 14$. ———, true velocity, $\sqrt{[2(P-p)/\rho]}$; ———— "velocity" based on wall static pressure, $\sqrt{[2(P-p_w)/\rho]}$; — · — · — "velocity" based on static pressure on matching surface, $\sqrt{[2(P-p_e)/\rho]}$.

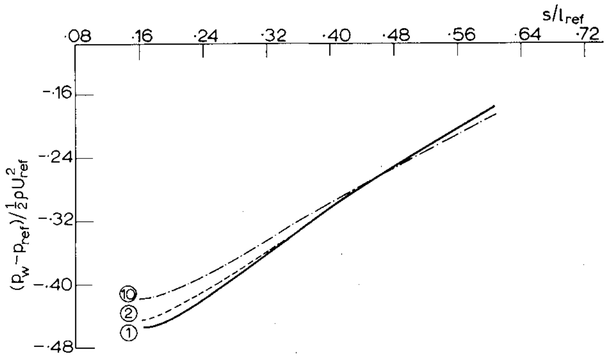

Fig. 8 Convergence of wall pressure in calculation for NACA 0012 airfoil. Curves are outputs after the number of iterations indicated.

region of rapid change. The need for a $\partial p/\partial y$ calculation in flows like that of Fig. 2 is best demonstrated by the velocity profile shown in Fig. 7. This figure shows the calculated velocity profile, the profile corresponding to the same total-pressure distribution and the assumption that $p = p_w$ everywhere, and the profile corresponding to the assumption $p = p_e$. A conventional boundary-layer calculation using either p_e or p_w would seriously misrepresent the shear applied to the turbulence and difficulties would arise in the definition of displacement thickness.

Figure 8 shows an external-flow case, the boundary layer on a NACA 0012 airfoil from the leading edge to about 60% chord, starting with a laminar flow calculation by Thwaites' method and switching to a turbulent flow at an assumed transition point. The present method is not necessary for such a simple flow and the figure is included merely to demonstrate the existence of the external-flow version of the method. The values of δ_0 and V_e downstream of 60% chord were guessed and the wake was ignored; these shortcomings are temporary.

Conclusions

The procedure just described is economical in time and storage and avoids all the difficulties associated with the breakdown of the boundary-layer approximation. It can be applied to almost any existing field method of shear-layer calculation and to almost any flow without recirculation. The extension to wakes and thus to full airfoil calculations is in progress, and a fuller report will be issued when this work is complete. The only obstacle to an extension to supercritical compressible flow is the greater complication of the inviscid calculation. No change of differencing within the shear layer should be needed, and the solution of Eq. (2) would be unaltered. Work is also in progress on flows with small regions of recirculation (separation bubbles); a singularity should not appear at the separation point because $\partial p/\partial x$ at the surface is a result of the shear-layer calculation, and a simplified version of the DUIT procedure[11] is being used to represent upstream influence via the velocity field as well as the pressure.

The extension of the procedure to unseparated three-dimensional flows is straightforward in principle, at least if spanwise gradients are of no greater order than streamwise gradients. In this case $p(x,y,z)$ would have to be guessed and stored, the new profiles of U, W, σ_{xy}, σ_{zy}, and V would be calculated at given (x,z) by the same numerical procedure as if $\partial p/\partial y$ were zero, and then the new profile of p would be calculated at the same (x,z) by integrating Eq. (2) with the additional (known) term $W\partial V/\partial z$ on the left-hand side. The shear layer calculation would be marched in x and z in the same way as if $\partial p/\partial y$ were zero, and the interaction with the inviscid flow would be calculated with $V_e(x,z)$ as input and $p_e(x,z)$ as output. The computing time and storage requirement would depend on the size of the region of significant $\partial p/\partial y$ and, though small compared to the needs of a full Navier Stokes program for this region, are likely to preclude development of a three-dimensional method by the present authors.

Acknowledgments

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